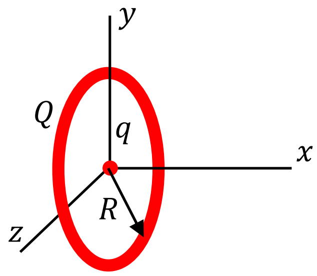
PHYS 102 Midterm Exam 1 Solution 2021-22-2

1. Charge $+Q$ is uniformly distributed around a thin ring of radius R which is fixed on the yz-plane with its center at the origin. A point charge $+q$ placed at the center of the ring. Use the coordinate system indicated in the figure to answer the following questions.



(a) (8 Pts.) What is the electric potential for $x > 0$?
(b) (8 Pts.) What is the electric field (vector) created by this distribution of charge for $x > 0$?
(c) (7 Pts.) If the charge $+q$ at the center of the ring is displaced along the positive x-axis a distance d from the origin while the ring is kept fixed, what would be the work done by the force displacing it?
(d) (7 Pts.) If the charge $+q$ is slightly displaced from the origin while the ring is kept fixed, it will accelerate along the x-direction. Given that its mass is m, what will be its speed in the limit $x \to \infty$?

Solution: (a) Because of the symetry of the charge distribution, the electric field will be in the $\hat{\mathbf{i}}$ direction. The x-component of the total electric field is $E_x = E_q + E_Q$. Electric field created by the point charge at the origin along the positive x-axis is

$$E_q = \frac{q}{4\pi\epsilon_0 x^2}.$$

For the ring of charge, calculating the electric potential is easier. We have

$$V_Q = \frac{1}{4\pi\epsilon_0}\int \frac{dQ}{\sqrt{x^2+R^2}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{x^2+R^2}}.$$

Electric field created by the ring can be found from the potential as

$$E_Q = -\frac{dV_Q}{dx} = \frac{Q}{4\pi\epsilon_0}\frac{x}{(x^2+R^2)^{3/2}}.$$

Therefore,

$$E_x = E_q + E_Q = \frac{q}{4\pi\epsilon_0 x^2} + \frac{Q}{4\pi\epsilon_0}\frac{x}{(x^2+R^2)^{3/2}} \quad \to \quad \vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{x^2} + \frac{Qx}{(x^2+R^2)^{3/2}}\right]\hat{\mathbf{i}}, \qquad x > 0.$$

(b)

$$V(x) = V_q(x) + V_Q(x) = \frac{q}{4\pi\epsilon_0 x} + \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{x^2+R^2}} \quad \to \quad V(x) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{x} + \frac{Q}{\sqrt{x^2+R^2}}\right), \qquad x > 0.$$

(c) By definition of potential, the work done by the electric force is $\Delta W = q(V'_0 - V'_d)$, where V'_0, V'_d denote potential created by the charged ring at points $x = 0$ and $x = d$ respectively. The work done by the force $\vec{\mathbf{F}}$ displacing the charge is $\Delta W_F = -\Delta W$. Hence

$$\Delta W_F = \frac{qQ}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{d^2+R^2}} - \frac{1}{R}\right).$$

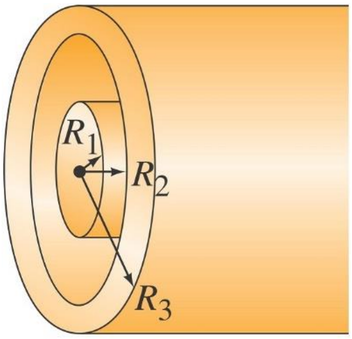
(d) By conservation of energy

$$\frac{1}{2}mv^2(x) + u(x) = \text{constant}.$$

This means

$$\frac{1}{2}mv^2(0) + u(0) = \frac{1}{2}mv^2(x\to\infty) + u(x\to\infty) \quad \to \quad \frac{1}{2}mv_\infty^2 = \frac{qQ}{4\pi\epsilon_0 R} \quad \to \quad v_\infty = \sqrt{\frac{qQ}{2\pi\epsilon_0 mR}}.$$

2. An infinite solid nonconducting cylinder of radius R_1 is uniformly charged with a charge density ρ. Coaxial with this cylinder is an infinite conducting cylindrical shell of inner radius R_2 and outer radius R_3, which has zero net charge.



(a) (25 Pts.) Find the electric field magnitude in regions $0 < r < R_1, R_1 < r < R_2$, $R_2 < r < R_3$, and $r > R_3$.

(b) (10 Pts.) Find the surface charge density on the inner surface of the cylindrical shell.

Solution: We apply Gauss's law to a cylinder of radius r and length L concentric with the solid nonconducting cylinder. The flux of the electric field through the cylinder is

$$\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \int_{\text{side}} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} + \int_{\text{top}} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} + \int_{\text{bottom}} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}}\,.$$

On the top and the bottom surfaces the normal to the surface is perpendicular to the electric field, which makes the last two integrals on the right hand side zero. On the side surface the normal is parall to the electric field. Hence,

$$\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \int_{\text{side}} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = (2\pi r L) E(r) = \frac{Q_{\text{enc}}}{\epsilon_0}.$$

The charge enclosed by the surface is $Q_{\text{enc}} = \rho V_{\text{enc}} = \rho \pi r^2 L$. Therefore, According to Gauss' law

$$(2\pi r L) E(r) = \frac{\rho \pi r^2 L}{\epsilon_0} \quad \rightarrow \quad E(r) = \frac{\rho r}{2\epsilon_0}\,, \qquad 0 < r < R_1\,.$$

If $R_1 < r < R_2$, the charge enclosed by the surface is $Q_{\text{enc}} = \rho V_{\text{enc}} = \rho \pi R_1^2 L$. Hence

$$(2\pi r L) E(r) = \frac{\rho \pi R_1^2 L}{\epsilon_0} \quad \rightarrow \quad E(r) = \frac{\rho R_1^2}{2\epsilon_0 r}\,, \qquad R_1 < r < R_2\,.$$

The region $R_2 < r < R_3$ is a conductor. Therefore $E(r) = 0\,,\ R_2 < r < R_3$.

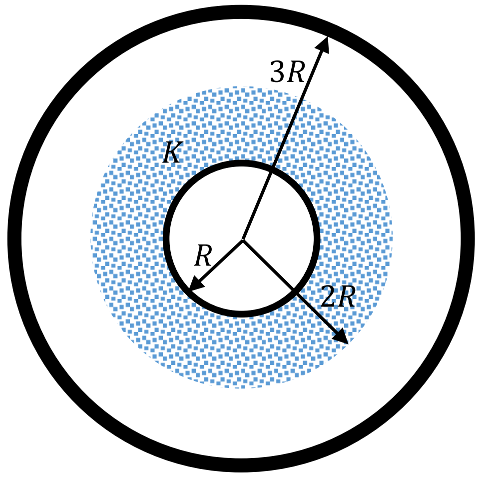
If $r > R_3$, the charge enclosed is again $Q_{\text{enc}} = \rho \pi R_1^2 L$. Therefore

$$E(r) = \frac{\rho R_1^2}{2\epsilon_0 r}\,, \qquad R_3 < r\,.$$

(b) For $R_2 < r < R_3$ we found $E(r) = 0$, meaning that $Q_{\text{enc}} = 0$. Since there is charge $Q_{\text{in}} = \rho \pi R_1^2 L$ on the inner cylinder, equal amount of negative charge must accumulate on the inner surface of the cylindrical shell of radius R_2 to make the total enclosed charge zero. Therefore

$$\sigma_{\text{in}} = -\frac{\rho \pi R_1^2 L}{2\pi R_2 L} \quad \rightarrow \quad \sigma_{\text{in}} = -\frac{\rho R_1^2}{2R_2}\,.$$

3. A spherical capacitor is made from a metal sphere of radius R and a concentric metal shell of inner radius $3R$. The region from $r = R$ to $r = 2R$ is filled with a dielectric material of dielectric constant K. A cross section of the capacitor is shown in the figure.



(a) (20 Pts.) What is the capacitance of this capacitor?
(b) (15 Pts.) If the capacitor is charged so that potential difference between the outer and inner conductor is $V(3R) - V(R) = V_0$, what is the potential difference between the surface of the dielectric and the inner conductor $V(2R) - V(R)$?

Solution: Assume that the metal sphere of radius R has charge Q on it. Using Gauss's law we find the electric field between the conductors as

$$\vec{\boldsymbol{E}}(r) = \frac{Q}{4\pi K\epsilon_0 r^2}\hat{\mathbf{r}}\,, \qquad R < r < 2R\,, \qquad \vec{\boldsymbol{E}}(r) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}\,, \qquad 2R < r < 3R\,.$$

Potential diference between the metal sphere and the inner surface of the metal shell is

$$|\Delta V| = \int_R^{2R} \frac{Q\,dr}{4\pi K\epsilon_0 r^2} + \int_{2R}^{3R} \frac{Q\,dr}{4\pi\epsilon_0 r^2} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{K}\int_R^{2R}\frac{dr}{r^2} + \int_{2R}^{3R}\frac{dr}{r^2}\right) \quad\rightarrow\quad |\Delta V| = \frac{Q}{8\pi\epsilon_0 R}\left(\frac{1}{K}+\frac{1}{3}\right) = \frac{Q(K+3)}{24\pi K\epsilon_0 R}\,.$$

Since, by definition, $Q = C\,|\Delta V|$, we find

$$C = \frac{24\pi K\epsilon_0 R}{K+3} = 24\left(\frac{K}{K+3}\right)\pi\epsilon_0 R\,.$$

(b) Given that

$$|\Delta V| = V(3R) - V(R) = \frac{Q(K+3)}{24\pi K\epsilon_0 R} = V_0 \quad\rightarrow\quad Q = \frac{24\pi K\epsilon_0 R}{K+3}V_0\,.$$

Since

$$|V(2R) - V(R)| = \int_R^{2R} \frac{Q\,dr}{4\pi K\epsilon_0 r^2} = \frac{Q}{8\pi K\epsilon_0 R} \quad\rightarrow\quad |V(2R) - V(R)| = \left(\frac{3}{K+3}\right)V_0\,.$$